GENERALIZED POLYNOMIAL APPROXIMATION

BY

J. BAK, D. LEVIATAN, D. J. NEWMAN AND J. TZIMBALARIO

ABSTRACT

We estimate the rate of covergence to functions in the spaces $L^p\,[0,1]$ and C[0,1] by polynomial of the form $\Sigma_\lambda a_\lambda x^\lambda$, where the λ's are positive real numbers and 0.

Introduction

We will use the following notation:

$$\Lambda = \{1, x^{\lambda_1}, \cdots, x^{\lambda_n}\} \quad 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$$

$$\|f\|_p = \left(\int_0^1 |f(t)|^p dt\right)^{1/p}$$

$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|$$

$$d_p(f;\Lambda) = \inf_{P_\Lambda \in [\Lambda]} \|f - P_\Lambda\|_p.$$

Our goal in this paper is to determine the degree of approximation possible to functions in the spaces $L^p[0,1]$ and $C[0,1]$ by "polynomials" P_Λ in the span of Λ. More specifically, we seek the approximation index I_p, that is, the smallest possible number $\eta_p(\Lambda)$ such that for any $f \in L^p$,

$$d_p(f;\Lambda) \leqq 2W_p(f;\eta_p).$$

Here W_p denotes the L^p modulus of continuity of f. (To simplify notation, we use the symbol L^∞ to denote the space of continuous functions.)

Recent research ([1], [6], [7]) has yielded the following results:

(A) If $2 \leqq p \leqq \infty$ and $\lambda_m \geqq 2m$, $m = 1, 2, \cdots n$, I_p is equivalent (equal within a constant factor) to

Received January 22, 1973

$$\exp\left(-2\sum_m \frac{1}{\lambda_m}\right).$$

(B) If $p = 2$ or $p = \infty$, I_p is equivalent to $\max_{\mathrm{Re}\, z=1} |B_p(z)/z|$, where $B_p(z)$ is the Blaschke product with zeros at $\lambda_m + 1/p$.

Note that (A) gives a handy formula for I_p in the L^p spaces, $2 \leqq p \leqq \infty$ under the growth condition $\lambda_m \geqq 2m$. (B) gives the complete answer in the special cases $p = 2$ or $p = \infty$. Furthermore, the rather complicated expression in (B) has a simple equivalent also in the "nonseparated" cases. That is, setting $\varepsilon_p = \max_{\mathrm{Re}\, z=1} |B_p(z)/z|$, we have

(1) $$C_2\left(\sum_m \lambda_m\right)^{-\frac{1}{2}} \leqq \varepsilon_p \leqq C_1\left(\sum_m \lambda_m\right)^{-\frac{1}{2}}, \qquad \text{for each } 1 \leqq p \leqq \infty,$$

as long as $\lambda_{m+1} - \lambda_m \leqq 2$. See [7].

Our conjecture is that ε_p is the correct formula (up to a constant factor) for the approximation index in all the L^p-spaces. The results we prove will show that it serves as an upper bound under very general conditions and also serves as a lower bound for $2 \leqq p \leqq \infty$ and under the added hypothesis $\lambda_m \geqq 2m$ also for $1 \leqq p < 2$. For future reference, then, we record the equivalent formula for ε_p also in the "separated" case. There, a comparison of Theorems (A) and (B) shows

(2) $$C_2 \exp\left(-2\sum_m \frac{1}{\lambda_m}\right) \leqq \varepsilon_p \leqq C_1 \exp\left(-2\sum_m \frac{1}{\lambda_m}\right)$$

as long as $\lambda_m \geqq 2m$ for all m. In fact (2) holds for each $1 \leqq p \leqq \infty$.

1. Upper bounds

THEOREM 1. *There exists a constant $A > 0$ such that for any $f \in L^p[0,1]$, there exists P_Λ with $\|f - P_\Lambda\|_p < 2\, W_p(f; A\varepsilon_p)$ if any one of the following conditions is satisfied*:

(a) $2 \leqq p \leqq \infty$
(b) $1 \leqq p < 2$, $\lambda_m \geqq m$ *for all* m
(c) $1 \leqq p < 2$, $\lambda_{m+1} - \lambda_m \leqq 1$ *for all* m.

Since $\lambda_{m+1} - \lambda_m \geqq 1$ for all m implies $\lambda_m \geqq m$ for all m, cases (b) and (c) form a pair of "separated" and "nonseparated" cases.

Our method of proof will consist of first approximating by an ordinary N-th degree polynomial and then reapproximating the monomials by Λ-polynomials.

This is the method introduced by Von Golitschek [9] and Leviatan [5] and recently used by Newman [6]. We will need the following lemmas:

LEMMA 1. *Suppose* $f \in L^p[-1,1]$ *with* $\|f'\|_p \leqq 1$. *Then there exists an* N*-th degree polynomial* $P_N(x)$ *such that*

$$\|f - P_N\|_p \leqq A_1/N \quad \textit{while} \quad |P_N'| \leqq A_2 N^{1/p}. \tag{3}$$

PROOF. For even periodic L^p-functions $g(\theta)$ on $[-\pi,\pi]$ with $\|g'\|_p \leqq 1$, the even trigonometric approximation $T_N(\theta)$ given by the Jackson kernel satisfies $\|g - T_N\|_p \leqq C_1/N$ while $|T_N'| \leqq C_2 N^{1/p}$. See [4, p. 2]. For $f \in L^p[-\frac{1}{2},\frac{1}{2}]$, we obtain (3) by making the usual transformation $x = \cos\theta$, $g(\theta) = f(\cos\theta)$. Finally, a second change of variables shows that (3) holds for the interval $[-1,1]$, as desired.

LEMMA 2. *For any* $p \geqq 1$ *and* $k \geqq 1$, $|B_p(k)| \leqq k^k \varepsilon_p^k$.

PROOF. The proof is given in [6] under the hypothesis that k is a positive integer, but the proof is identical as long as $k \geqq 1$.

LEMMA 3. *In each of the cases* (a), (b) *and* (c) *and for any positive integer* k, *there exists a* Λ*-polynomial* P_Λ *such that*

$$\|x^k - P_\Lambda(x)\|_p \leqq B^{k+1}(k+1)^{k+1}\varepsilon_p^{k+(1/p)},$$

B an absolute positive constant.

PROOF OF LEMMA 3. Here we must consider each case separately:

Case a. $2 \leqq p \leqq \infty$.

We begin by noting that the L^p distance of x^k to Λ is given by

$$\sup \int_0^1 x^k g(x)dx$$

where the supremum is taken over all functions $g \in L^q\,[0,1]$ of norm one and s.t. $\int_0^1 x^{\lambda_m} g(x)dx = 0$ for all m. Given such a function $g(x)$, we set

$$F(z) = \int_0^1 x^{z-1/p}\, g(x)dx$$

and note that $F(z)$ is an analytic function in the right half-plane with zeros at the points $\lambda_m + 1/p$, so that

$$F(z) = B_p(z)\,G(z).$$

To estimate $G(z)$, we rewrite $F(z)$ as an integral over the half-line

$$F(z) = \int_0^\infty e^{-tz}e^{-t/q}g(e^{-t})dt = B_p(z)G(z).$$

Noting then that $\| e^{-t/q}g(e^{-t}) \|_q$ on $[0,\infty) = 1$ and $| B(iy) | = 1$ for all y, we can apply Young's inequality [10, p. 316] to conclude

$$\int_{-\infty}^{\infty} | G(iy) |^p \, dy \leqq 2\pi.$$

Finally, since $G(z)$ is analytic in the right half-plane

$$G(k) = \frac{k}{\pi i} \int_I G(z)dz/(z-k)(z+k)$$

where I is the imaginary axis and by Hölder's inequality

$$| G(k) | \leqq (2\pi)^{1/p}(k/\pi) \int_I | dz | / | z-k |^q \, | z+k |^q)^{1/q}$$

and a direct estimate shows

$$| G(k) | \leqq (2/k)^{1/p}.$$

Recalling that the L^p distance of x^k to Λ is bounded by $\sup_F F(k+1/p)$, we have for some P_Λ,

$$\| x^k - P_\Lambda \|_p \leqq | B_p(k+1/p)G(k+1/p) | \leqq 2 | B_p(k+1/p) |$$

and an application of Lemma 2 completes the proof.

Case b. $1 \leqq p < 2$, $\lambda_m \geqq m$.

Our proof here rests on the following inequality derived in [1]. (Interestingly, it was used there to obtain a *lower* bound in the conjugate spaces.)

Definition. Suppose $F(z) = \int_0^\infty e^{-tz}f(t)dt$,

$$\| F(z) \|_{A_p} = \| f(t) \|_p \text{ on } [0,\infty].$$

Proposition.

$$\| B(z)/(z+k) \|_{Ap} \leqq 3 \textit{ for } 1 \leqq p \leqq 2.$$

The proposition is proven in [1, p. 452] for $\lambda_m \geqq 2m$ but is evidently true for $\lambda_m \geqq \delta m$, where $\delta > 0$ is arbitrary but fixed. While the proof there involves $k = 1$, the arguments are quite general.

Now, by a partial fraction decomposition

$$B(z)/(z+k) = B(-k)/(z+k) + \Sigma A_\lambda/(z+\lambda).$$

Thus

$$B(z)/(z+k) = B(-k)\int_0^\infty e^{-tz}(e^{-tk} - \Sigma C_\lambda e^{-t\lambda})dt$$

so that according to the above proposition,

$$\| e^{-tk} - \Sigma C_\lambda e^{-t\lambda} \|_p \text{ on } [0,\infty] \leqq 3 |B(-k)|^{-1} = 3|B(k)|.$$

The standard transformation $e^{-t} = x$ shows then that

$$\| x^{k-1/p} - \Sigma C_\lambda x^{\lambda-(1/p)} \|_p \text{ on } [0,1] \leqq 3|B(k)|$$

or, equivalently,

$$\| x^k - P_\Lambda(x) \|_p \leqq 3|B_p(k+1/p)|$$

so that again by Lemma 2, the proof is complete.

Case c. $1 \leqq p < 2$, $\lambda_{m+1} - \lambda_m \leqq 1$.

Our proof here rests on the ability to approximate x^{k+1} in the uniform norm and the explicit representation for ε_p given in (1).

We begin by considering, for any k, the set of monomials $\{x^{\mu_m}\}$, where $\mu_m = (k + 1/k)\lambda_m$. If we set

$$B_p^*(z) = \prod_m \frac{z - (\mu_m + 1/p)}{z + (\mu_m + 1/p)} \text{ and } \eta_p = \max_{\mathrm{Re}\, z = 1} \left| \frac{B_p^*(z)}{z} \right|,$$

we have by Case (a)

$$\| t^{k+1} - \Sigma C_m t^{\mu_m} \|_\infty \leqq 2(k+1)^{k+1}\eta_\infty^{k+1}$$

for some choice of coefficients $C_0, C_1, \cdots, C_n$. Furthermore since $\|f(t^\alpha)\|_\infty = \|f(t)\|_\infty$ for any $\alpha > 0$, we can make the change of variables $t^{k+1} = x^k$ and conclude

$$\| x^k - \Sigma C_m x^{\lambda_m} \|_p \leqq \| x^k - \Sigma C_m x^{\lambda_m} \|_\infty \leqq 2(k+1)^{k+1}\eta_\infty^{k+1}$$

for all $p \geqq 1$.

Finally, since both of the sets $\{x^{\lambda_m}\}$ and $\{x^{\mu_m}\}$ have exponents which are separated by less than two, we can apply (1) to conclude

$$\varepsilon_p \geqq B_1(\Sigma\lambda_m)^{-\frac{1}{2}} \geqq B_2(\Sigma\mu_m)^{-\frac{1}{2}} \geqq B_3\eta_\infty$$

so that

$$\left\| x^k - \Sigma C_m x^{\lambda_m} \right\|_p \leqq B^{k+1}(k+1)^{k+1}\varepsilon_p^{k+1}$$

and the proof is complete.

PROOF OF THEOREM 1. Let f be such that $\|f'\|_p \leqq 1$; we will show the existence of a polynomial P_Λ such that $\|f - P_\Lambda\| \leqq A \cdot \varepsilon_p$ (the reduction to this special case is contained e.g. in [1]).

Let $P_N(x)$ be the N-th degree approximator guaranteed by Lemma 1 and recall that

$$(4) \qquad |Q'_N| \leqq 1 \Rightarrow |Q_N'^{(k)}(0)| \leqq N^{k-1} \qquad [8, \text{p. } 226].$$

Thus if

$$P_N(x) = \Sigma a_k x^k, \quad |a_k| \leqq A_2 N^{k-1+1/p}/k!.$$

Finally taking $P_\Lambda(x) = \Sigma a_k R_k(x)$ where R_k is the best Λ-approximator to x^k we have by Lemma 3,

$$\|f - P_\Lambda\|_p \leqq \|f - P_N\|_p + \|P_N - P_\Lambda\|_p$$

$$\leqq \frac{A_1}{N} + \sum_{k=1}^{N} \frac{N^{k-1+1/p}A_3^k(k+1)^{k+1}\varepsilon_p^{k+1/p}}{k!}.$$

Choosing $N = \dfrac{1}{6A_3 \cdot \varepsilon_p}$,

$$\|f - P_\Lambda\|_p \leqq A_4\varepsilon_p + \varepsilon_p A_5 \sum_{k=1}^{\infty} k \cdot \left(\frac{2e}{6}\right)^k$$

and the proof is complete.

REMARK. As we mentioned in the proof of Case (b) the methods used could be applied as well under the somewhat weaker condition $\lambda_m \geqq \delta m$, $\delta > 0$. Nevertheless, not all cases are covered by Theorem 1 and one may wonder whether under some unusual conditions I_p may actually be much larger then ε_p. In the following sense this is impossible.

Proceeding as in the proof of Case (a), one could show using rough estimates that for any $p \geqq 1$ and positive integer k,

$$\|x^k - P_\Lambda(x)\|_p \leqq A(k+1)^{k+1}\varepsilon_p^{k+1/p}|\log \varepsilon_p|^{1/p}$$

for some P_Λ. This result may then be used as in the proof of Theorem 1 to show that

$$I_p \leqq A\varepsilon_p |\log \varepsilon_p|^{1/p}$$

in all L^p-spaces, and with no restrictions on $\lambda_1, \lambda_2, \cdots, \lambda_n$.

2. Approximation to differentiable functions

A second theorem of Jackson [4, p. 12] states that for any $f \in C^1[0,1]$ there exists an N-th degree polynomial P_N such that

$$\|f - P_N\|_\infty \leqq AW(f';1/N)/N.$$

Stated equivalently in terms of certain basic functions $f \in C^1[0,1]$ with $|f''| \leqq 1$, the theorem asserts that there exists P_N such that $\|f - P_N\| \leqq A/N^2$. Corresponding results hold for functions in $C^j[0,1]$, ($1/N^2$ is replaced by $1/N^{j+1}$), and in all the L^p-spaces as well. We wish to generalize these theorems in the direction of Muntz-approximation that is, we will prove

THEOREM 2. *Let ε_p be as before and assume the hypothesis of Theorem 1 is satisfied. Furthermore, supplement Λ to include the monomials $x, x^2, \cdots, x^j$. Then for any j-times differentiable $f \in L^p[0,1]$, there exists P_Λ such that*

$$\|f - P_\Lambda\|_p \leqq A\varepsilon_p^j W_p(f;\varepsilon_p).$$

REMARKS. To simplify notation, we will deal with the case $j = 1$ although the method generalizes easily. Also, we will confine ourselves to differentiable functions such that $|f''|_p \leqq 1$. We then seek to demonstrate the existence of a function P_Λ such that

$$\|f - P_\Lambda\|_p \leqq A\varepsilon_p^2$$

(the reduction to this special case and a proof under the hypotheses $\lambda_m \geqq 2m$, $p \geqq 2$ are contained in [1]).

PROOF OF THEOREM 2. We first use Jackson's method to approximate f by an N-th degree polynomial. Since $|f''|_p \leqq 1$, by (3) we can find Q_N s.t.

$$\|f' - Q_N'\|_p \leqq 1/N \text{ with } |Q_N''| \leqq N^{1/p}.$$

But then $N(f - Q_N)$ has a bounded derivative (L^p) and we can find R_N s.t.

$$\|N(f - Q_N) - R_N\|_p \leqq \frac{1}{N} \text{ with } |R_N'| \leqq N^{1/p}.$$

Taking $P_N = Q_N + R_N/N$, we have

$$\|f - P_N\| \leqq 1/N^2.$$

Also if $P_N(x) = \Sigma a_k x^k$, $|a_k| \leqq AN^{k-2+1/p}/k!$ for $k \geqq 2$.

This follows from the two inequalities $|Q_N''| \leqq N^{1/p}$ and $|R_N'/N| \leqq N^{-1+1/p}$ and (4).

Once again, reapproximating $P_N(x)$ by $P_\Lambda(x) = \sum a_k Q_k(x)$ where Q_k is the best Λ-approximator to x^k, we obtain by Lemma 3

$$\|f - P_\Lambda\| \leqq \|f - P_N\| + \|P_N - P_\Lambda\|$$

$$\leqq A/N^2 + \sum_{k=2}^{\infty} N^{k-2+1/p} B^{k+1} (k+1)^{k+1} \varepsilon_p^{k+1/p}/k!.$$

Choosing $N = 1/6\, B\varepsilon_p$ as before we obtain

$$\|f - P_\Lambda\| \leqq C\varepsilon_p^2$$

and the proof is complete.

3. Lower Bounds

THEOREM 3. *There exists a function $f \in L^p[0,1]$ such that $d_p(f,\Lambda) \geqq A\, W_p(f;\varepsilon_p)$ where A is an absolute positive constant (independent of f and Λ) if either $2 \leqq p \leqq \infty$ or $1 \leqq p < 2$ and $\lambda_k \geqq 2k$ for all k.*

PROOF. First assume $2 \leqq p \leqq \infty$. Our proof is based on some ideas and results from [1, 6]. Let $I_p = \sup_{\|f'\|_p \leqq 1} d_p(f,\Lambda)$, then I_p has a useful representation

$$(5) \qquad I_p = \sup_{g \in L^q[0,1]} \|G\|_q / \|g\|_q \text{ where } g \perp P_\Lambda \text{ and } G(x) = \int_0^x g(t)dt.$$

Given $g \in L^q[0,1]$ such that $g \perp P_\Lambda$ set

$$F(z) = \int_0^1 t^{z-1/p} g(t)dt.$$

Then $F(z)$ is analytic for $\operatorname{Re} z > 0$ and vanishes for $z = \lambda_i + 1/p$ $i = 0, 1, \cdots, n$. Also integration by parts yields

$$F(z) = -\left(z - \frac{1}{p}\right) \int_0^1 t^{z-(1/p)-1}\, G(t)dt$$

so that

$$\frac{F(z+1)}{z + (1/q)} = -\int_0^1 t^{z-(1/p)} G(t)dt\,.$$

Following [1, p. 451] we make a change of variable to obtain

$$F(z) = \int_0^\infty e^{-zt} h(t)dt$$

and

$$\frac{F(z+1)}{z+(1/q)} = \int_0^\infty e^{-zt}H(t)dt$$

where $h(t) = g(e^{-t})e^{-t/q}$ and $H(t) = G(e^{-t})e^{-t/q}$ so that $\| h \|_q$ on $[0,\infty) = \| g \|_q$ on $[0,1]$ and $\| H \|_q$ on $[0,\infty) = \| G \|_q$ on $[0,1]$. Since

$$\frac{F(iy+1)}{iy+(1/q)}$$

is the Fourier transform of $H(t)$ it follows by Young's inequality that

$$\left\| \frac{F(iy+1)}{iy+(1/q)} \right\|_p \leqq (2\Pi)^{1/p} \| H \|_q. \tag{6}$$

Now $F(z)$ may be factorized into $F(z) = B_p(z)A(z)$ where $B_p(z)$ is the Blaschke product vanishing at $z = \lambda_i + 1/p \quad i = 0,1,\cdots,n$ and $A(z)$ is analytic in the right half-plane. Hence

$$\left\| \frac{F(iy+1)}{iy+(1/q)} \right\|_p = \left(\int_{-\infty}^{\infty} | A(iy+1) |^p L(y)^{p/2} dy \right)^{1/p} \tag{7}$$

where

$$L(y) = \frac{1}{y^2 + \frac{1}{q^2}} \prod_{i=0}^{n} \frac{y^2 + \left(\lambda_i - 1 + \frac{1}{p}\right)^2}{y^2 + \left(\lambda_i + 1 + \frac{1}{p}\right)^2},$$

and we will use this to estimate

$$\left\| \frac{F(iy+1)}{iy+(1/q)} \right\|_p$$

from below. We do this in much the same way as [7]. Let $\gamma_i = \lambda_i + 1/p \quad i=0,\cdots,n$. Then for $y \geqq 0$

$$\frac{L'(y)}{L(y)} = \sum_{i=0}^{n} \left[\frac{2y}{y^2 + (\gamma_i - 1)^2} - \frac{2y}{y^2 + (\gamma_i + 1)^2} \right] - \frac{2y}{y^2 + \frac{1}{q^2}} \tag{8}$$

$$= 8y \sum_{i=0}^{n} \frac{\gamma_i}{[y^2 + (\gamma_i - 1)^2][y^2 + (\gamma_i + 1)^2]} - \frac{2y}{y^2 + \frac{1}{q^2}}$$

$$\geqq - \frac{2y}{y^2 + \frac{1}{q^2}} \geqq -q.$$

Let $t \geqq 0$ be the point where $L(y)$ assumes its maximum and put $M = L(t)^{1/2}$. Then integrating (8) from t to $t+u$, $u > 0$, we obtain

$$\log\frac{L(t+u)}{M^2} \geqq -qu \text{ or } L(t+u) \geqq M^2 e^{-qu}$$

for all $u > 0$. In particular

$$L(y) \geqq M^2 e^{-q}, \qquad t \leqq y \leqq t+1. \tag{9}$$

Now choose

$$F(z) = \frac{4z^2 s^2 B_p(z)}{(z+s)^2(z+1-it)}$$

where $s = \sqrt{1+t^2}$. This function has been used by Newman [6] for the case $p = \infty$. For this $F(z)$

$$A(z) = \frac{4z^2 s^2}{(z+s)^2(z+1-it)}.$$

From (7) and (9)

$$\left\|\frac{F(iy+1)}{iy+(1/q)}\right\|_p \geqq Me^{-q/2}\left(\int_t^{t+1}\left(\frac{4|1+iy|^2 s^2}{|1+iy+s|^4|1+iy+1-it|}\right)^p dy\right)^{1/p}$$

$$= Me^{-q/2}\left(\int_t^{t+1}\left(\frac{4(1+y^2)^2 s^4}{((1+s)^2+y^2)^4(4+(y-t)^2)}\right)^{p/2} dy\right)^{1/p}.$$

Now for $t \leqq y \leqq t+1$

$$\frac{(1+y^2)s^2}{[(1+s)^2+y^2]^2} = \frac{(1+y^2)(1+t^2)}{\left[\left(1+\sqrt{1+t^2}\right)^2+y^2\right]^2}$$

$$\geqq \frac{(1+t^2)^2}{\left[1+2\sqrt{1+t^2}+1+t^2+(1+t)^2\right]^2}$$

$$\geqq \frac{(1+t^2)^2}{[6(1+t^2)]^2} = \frac{1}{36}.$$

Therefore

$$\left\|\frac{F(iy+1)}{iy+(1/q)}\right\|_p \geqq \frac{M}{18}e^{-q/2}\left(\int_t^{t+1}\frac{dy}{[4+(y-t)^2]^{p/2}}\right)^{1/p} \tag{10}$$

$$= \frac{Me^{-q/2}}{18}\left(\int_0^1 \frac{dy}{(4+y^2)^{p/2}}\right)^{1/p}$$

$$= CM.$$

Evidently M is equivalent to ε_p thus (6) and (10) imply

$$\| H \|_q \geqq K\varepsilon_p. \tag{11}$$

We now find an absolute upper bound for $\| h \|_q$ $1 < q < 2$ and once this is done our theorem follows by (5) and (11).

To this end we introduce the following notation of Bak and Newman [1, p. 451]. Denote by $\|f\|_{A_2}$ the L^q-norm of the inverse Fourier transform of f. Newman [6, (14)] proved that

$$\| F(iy) \|_{A_1} \leqq 6. \tag{12}$$

Also by Parseval's equality and Newman [6, (12)]

$$\begin{aligned} \| F(iy) \|_{A_2} &= \sqrt{2\pi} \, \| F(iy) \|_2 \\ &= \sqrt{2\pi} \int_{-\infty}^{\infty} |F(iy)|^2 dy^{\,1/2} \\ &\leqq \sqrt{2}\pi. \end{aligned} \tag{13}$$

Combining (12) and (13) it follows that $\| F(iy) \|_{Aq}$ is bounded by an absolute constant for all $1 < q < 2$ and this completes the proof for $2 \leqq p \leqq \infty$. Thus let us assume $1 \leqq p < 2$.

Our proof will be based on two lemmas. The first one assures us that the monomial $x^{1-1/p}$ can be approximated (L^p) to within a constant times ε_p by the derivative of some Λ-polynomial.

LEMMA 1. *Let* $\Lambda' = \{x^{\mu_1}, x^{\mu_2}, \cdots, x^{\mu_{n-1}}\}$ *where* $\mu_k = \lambda_{k+1} - 1$ *then*

$$d_p(x^{1-1/p}, \Lambda') \leqq A_1\varepsilon_p.$$

PROOF OF LEMMA 1. According to Lemma 3b of Theorem 1,

$$d_p(x^{1-1/p}, \Lambda') \leqq \prod_{k=1}^{n-1} \left| \frac{\mu_k - 1 + 1/p}{\mu_k + 1 + 1/p} \right| = \prod_{2}^{n} \frac{\lambda_k - 2 + 1/p}{\lambda_k + 1/p}$$

$$\leqq \exp\left(-2 \sum_{k=3}^{n} 1/\lambda_k \right) \leqq A_1\varepsilon_p$$

by the left-hand inequality in (2).

LEMMA 2. $d_p(x^{2-1/p}; \Lambda) \geqq A_2\varepsilon_p^2$.

PROOF OF LEMMA 2. Again, we will use the fact that

$$d_p(f;\Lambda) \geqq \left| \int_0^1 \phi(x)f(x)dx \right|$$

for any $\phi \in L_q$ such that $\phi \perp \Lambda$ and $\| \phi \|_q \leqq 1$.

Choose ϕ so that

$$\int_0^1 \phi(x)x^z dx = \frac{zB_p(z+1/p)}{5(z+2)^3}.$$

Since $B_p(\lambda + (1/p)) = 0, \phi \perp \Lambda$. We wish to show $\| \phi \|_q \leqq 1$. Note, then, that if we make the transformation $x = e^{-t}$ and set $z = iy - 1/p$, we have

$$\int_0^\infty \phi(e^{-t})e^{-t/q}e^{-ity}dt = \frac{(iy-1/p)B_p(iy)}{5(iy+2-1/p)^3}$$

so that

$$\phi(e^{-t})e^{-t/q} = \int_{-\infty}^{\infty} \frac{(iy-1/p)B_p(iy)e^{ity}}{5(iy+2-(1/p))^3}dy$$

and by Young's inequality

$$\| \phi(e^{-t})e^{-t/q} \|_q \text{ on } [0,\infty] \leqq 1.$$

Returning to the unit interval, we have shown that $\| \phi \|_q \leqq 1$. Thus

$$d_p(x^{2-1/p};\Lambda) \geqq \int_0^1 \phi(x)x^{2-1/p}dx = \frac{\left(2-\frac{1}{p}\right)B_p(2)}{5\left(4-\frac{1}{p}\right)^3}.$$

The proof follows then by noting that by the hypothesis on $\{\lambda_k\}$

$$|B_p(2)| \geqq A_1 \exp\left(-4\Sigma\frac{1}{\lambda}\right) \geqq A_2\varepsilon_p^2$$

the latter inequality following from the right-hand inequality in (2).

Proof of Thoerem 3. We will exhibit a function f such that

$$\|f'\|_p \leqq 1, \text{ while } d_p(f;\Lambda) \geqq A\varepsilon_p.$$

Choose $a_1, a_2, \cdots, a_n$ so that $\Sigma a_k x^{\lambda_k - 1}$ is the best Λ' approximation to $x^{1-1/p}$ and let

$$f(x) = \left[x^{2-1/p} - \Sigma\frac{(2-1/p)a_k}{\lambda_k}x^{\lambda_k}\right] \Big/ 2A_1\varepsilon_p.$$

By Lemma 1,

$$\|f'\|_p \leqq 1$$

while on the other hand

$$d_p(f;\Lambda) = d_p\left(\frac{x^{2-(1/p)}}{2A_1\varepsilon_p};\Lambda\right) \geqq \frac{A_2}{2A_1}\varepsilon_p$$

by Lemma 2, and the proof is complete.

References

1. J. Bak and D. J. Newman, *Müntz-Jackson theorems in L^p and C*, Amer. J. Math. **154** (1972), 437–457.

2. J. Bak and D. J. Newman, *Müntz-Jackson theorems in L^p, $p < 2$*, J. Approximation Theory, to appear.

3. T. Ganelius and S. Westlund, *Degree of approximation in Müntz's theorem*, Proc. Int. Conf. on Math. Anal., Jyväskylä, Finland, 1970.

4. D. Jackson, *The Theory of Appromixation*, A. M. S. Colloquium Pubs., Vol. XI, New York, 1930.

5. D. Leviatan, *On the Jackson-Müntz theorem*, J. Approximation Theory, to appear.

6. D. J. Newman, *A general Müntz-Jackson theorem*, Amer. J. Math., to appear.

7. D. J. Newman, *Müntz-Jackson Theorem in L^2*, J. Approximation Theory, to appear.

8. A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, New York, 1963.

9. M. Von Golitschek, *Erweiterung der Approximationssatze von Jackson im Sinne von Ch. Müntz II*, J. Approximation Theory **3** (1970), 72–85.

10. A. Zygmund, *Trigonometrical Series*, Monografje Mathematyczne, Tom V, Warszawa-Lwow, 1935.

City College of New York
New York, U. S. A.
Tel Aviv University
Tel Aviv, Israel
and
Yeshiva University
New York, U. S. A.